

# Shopping Round Solutions

May 17, 2026

LAMT 2026

1. [**\$3, 4**] Find the sum of the digits of

$$67 + 6767 + 676767 + 67676767 + 6767676767.$$

**Solution:**  $\boxed{42}$

To make life easier factor out 67:

$$67(1 + 101 + 10101 + 1010101 + 101010101) = 67 \cdot 102030405,$$

which is easy to multiply out. The final answer is 6836037135, which has digit sum  $\boxed{42}$ .

2. [**\$4, 5**] Tired of all the 67 brainrot, Albert decides to bring back the early 2010s dank memes (which are obviously better), and writes  $9 + 10 = 21$  on a board. However, Vicky, who is a newgen, is confused and reasons that Albert forgot to specify the base of the numbers he wrote. Vicky changes Albert's equation to the factual equation  $9_a + 10_a = 21_b$ , where  $a$  and  $b$  are positive integers (so 9 and 10 are read in base  $a$  and 21 in base  $b$ ). Find the smallest possible value of  $a + b$ .

**Solution:**  $\boxed{19}$

We are given that  $9_a + 10_a = 21_b$ . We can translate each quantity to base 10, which gives us  $9_a = 9$ ,  $10_a = a$ , and  $21_b = 2b + 1$ .

Now our equation reads  $9 + a = 2b + 1$ , which we can rearrange to get  $b = \frac{8+a}{2}$ . We know from  $9_a$  that  $a > 9$ , and  $21_b$  that  $b > 2$ . Additionally, we are told that both  $a$  and  $b$  are integers. Since 10 is the smallest possible value of  $a$ , we can plug that in to get  $a = 10$ ,  $b = 9$ . Furthermore, since  $\frac{8+a}{2}$  is strictly increasing,  $b$  will only become larger.

So our answer is  $a + b = 10 + 9 = \boxed{19}$ .

3. [**\$5, 7**] There are some pennies, nickels, dimes, and quarters sitting on a table, with seven coins in total. If Luke takes a set of 5 coins from the table, the largest possible total he can attain is  $M$ , and the smallest is  $m$ . Given that  $M - m = 14$ , find the largest possible total value of the seven coins, in cents.

**Solution:**  $\boxed{56}$

Number the coins  $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7$ . Note that when we switch from the smallest to the largest possible value we increase by  $a_7 + a_6 - a_2 - a_1 = 14$ . Note that this value is  $4 \pmod{5}$ , thus  $a_1 = 1$ . Then  $a_7 + a_6 - a_2 = 15$ . It is easy to see this implies  $a_7 = 10$ ,  $a_6 = 10$ , and  $a_2 = 5$ . Then  $a_3$ ,  $a_4$ , and  $a_5$  are at most 10. Thus the answer is  $1 + 5 + 10 + 10 + 10 + 10 + 10 = \boxed{56}$ .

4. [**\$5, 8**] Order the following from least to greatest:

$$1) 57! \quad 2) 5^{7!} \quad 3) (5^7)! \quad 4) (5!)^7.$$

Enter your answer as a permutation of the digits 1234. For instance, if you think  $5^{7!} < 57! < (5^7)! < (5!)^7$  your answer will be 2134.

**Solution:**  $\boxed{4123}$

First we will show  $(5!)^7 < 57!$ . We have

$$(5!)^7 = 120^7 < 1000^7 = 10^{21} < 10^{47} < 11 \cdot 12 \dots 57 < 57!.$$

Next we show  $5^{7!} > 57!$ . We have

$$5^{7!} = 5^{5040} = 125^{1680} > 125^{57} > 57!.$$

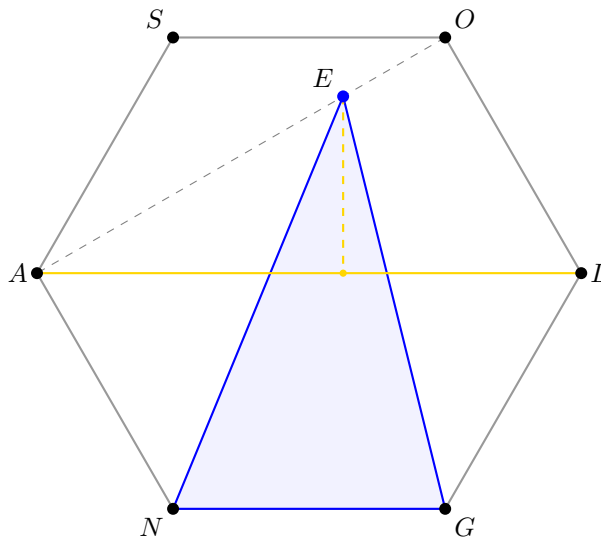
Finally we show  $(5^7)! > 5^{7!}$ . We know  $5^7$  is at least 20000 so

$$(5^7)! > 10000^{10000} > 125^{10000} > 125^{1680} = 5^{7!}.$$

Thus our order is  $(5!)^7 < 57! < 5^{7!} < (5^7)!$ , so the answer is  $\boxed{4123}$ .

5. **[\$6, 10]** Regular hexagon  $LOSANG$  has side length 4. Point  $E$  lies on diagonal  $OA$  such that  $OE = \sqrt{3}$ . Find the area of  $GEN$ .

**Solution:**  $\boxed{7\sqrt{3}}$



Since the side length is 4, using  $30-60-90$  ratios, we get that  $AO = 4\sqrt{3}$ . Since  $OE = \sqrt{3}$ , we see that  $E$  is  $\frac{3}{4}$  of the way from  $A$  to  $O$ .

The height from  $O$  to  $AL$  is  $2\sqrt{3}$  using  $30-60-90$  ratios again. The height from  $E$  is then  $\frac{3}{4}$  of this. Combining with the height from  $A$  to  $NG$ , we get that the height from  $E$  to  $NG$  is

$$2\sqrt{3} \cdot \left(1 + \frac{3}{4}\right).$$

The answer is then

$$4 \cdot 2\sqrt{3} \cdot \left(1 + \frac{3}{4}\right) \cdot \frac{1}{2} = \boxed{7\sqrt{3}}.$$

6. **[\$6, 11]** Find the unique ordered pair  $(m, n)$  of positive integers  $m < n$  for which  $m$  and  $n$  both have exactly 16 positive divisors, and  $m + n = 856$ .

**Solution:**  $\boxed{(216, 640)}$

Note that  $m$  and  $n$  have the same parity. If they are both odd they will be pretty large. Thus  $m$  and  $n$  are likely both even. Additionally note that  $3 = \nu_2(856) = \nu_2(m + n)$ , so likely one of  $m$  and  $n$  has  $\nu_2$  value equal to 3.

Then the other has  $\nu_2$  value greater than 3, but it must be 1 less than a power of 2. So we try  $\nu_2 = 7$ . Then we have to solve  $8a + 128b = 856$ . It is easy to see  $a = 27$  and  $b = 5$  is a solution which works. Then  $(m, n) = \boxed{(216, 640)}$ .

7. [**\$7, 13**] Patrick puts each integer from 1 through 9 in a distinct cell of a  $3 \times 3$  grid. He then starts in the cell labeled 1, and every second he travels to the neighboring cell with the largest value. Suppose his path is  $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 9$ . Find the number of ways Patrick could have filled in the grid.

**Solution:** 40

For Patrick to travel from  $1 \rightarrow 3$ , the only other number adjacent to 1 could be 2. This implies that the 1 is in a corner, giving 4 options. There are 2 ways to choose the location of the 3. From here, easy casework gives us 5 cases:

1	3	5
2	4	7
6	8	9

1	3	5
2	4	7
8	6	9

1	3	4
2	5	7
8	6	9

1	3	4
2	5	6
8	7	9

1	3	4
2	5	6
9	8	7

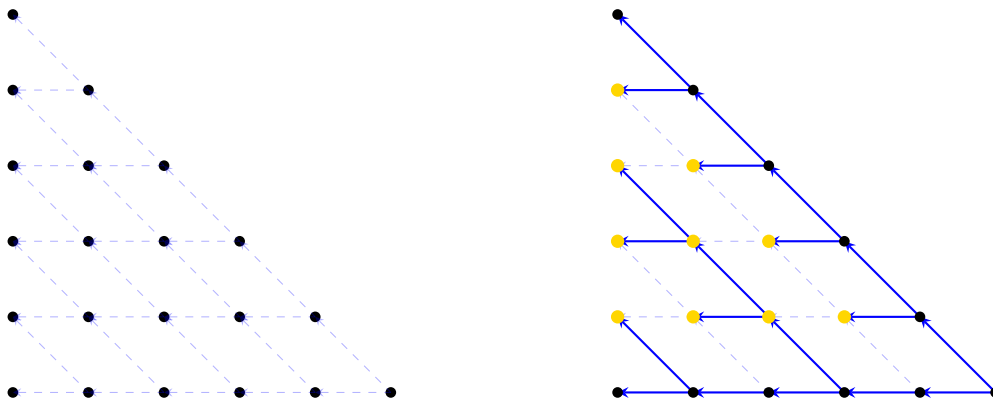
In total, our answer is  $4 \cdot 2 \cdot 5 = \boxed{40}$ .

8. [**\$7, 14**] An island is located at each point  $(x, y)$  for which  $x$  and  $y$  are positive integers satisfying  $x + y \leq 7$ . Pierre builds two types of one way bridges between islands:

- from each point  $(x + 1, y)$ , a one way bridge may be built to  $(x, y)$ , or
- from each point  $(x + 1, y - 1)$ , a one way bridge may be built to  $(x, y)$ .

Find the number of bridge arrangements satisfying the following: for every island in  $S$ , there is exactly one directed path from  $(6, 1)$  to that island that does not reuse any bridge.

**Solution:** 1024



By shifting the lattice and turning it into a triangular lattice, we basically have 6 rows, with  $k$  points in the  $k$ th row. For the 11 points on the outside of the lattice, they only have one option for who their parent is. For the remaining 10 points (in gold), they have two options. The answer is then  $2^{10} = \boxed{1024}$ .

9. [**\$8, 17**] Let  $s_b(n)$  denote the sum of the digits of  $n$  when written in base  $b$ . Find the smallest positive integer  $n$  for which  $\frac{s_2(n)}{s_4(n)} = \frac{5}{9}$ . Express your answer in base 10.

**Solution:** 171

We will consider the digits in base 2 two at a time, and the digits in base 4 one at a time. Note that since  $4 = 2^2$ , these are equivalent.

The digit 0 in base 4 is 00 in base 2, the digit 1 is 01, the digit 2 is 10, and the digit 3 is 11. Having 0s won't do anything, so let  $a$ ,  $b$ , and  $c$  be the number of 1/01s, 2/10s, and 3/11s, respectively. Then we have

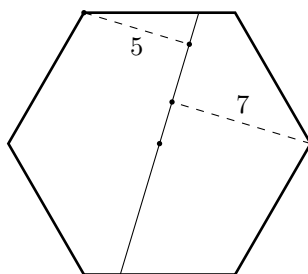
$$\frac{5}{9} = \frac{a + b + 2c}{a + 2b + 3c}.$$

Expanding and rearranging, we get

$$5a + 10b + 15c = 9a + 9b + 18c \implies b = 4a + 3c.$$

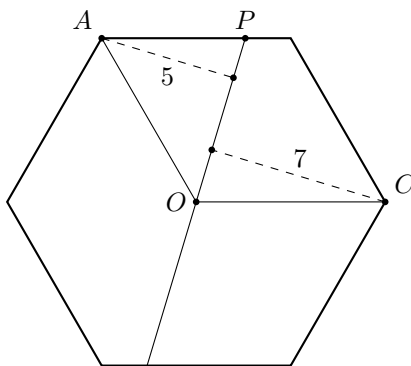
We want to minimize the number of digits, which means we should try to minimize  $a + b + c$ . By inspection, the smallest possible value of  $a + b + c$  is 4, when  $a = 0$ ,  $b = 3$ , and  $c = 1$ . The smallest possible value is achieved with  $2223_4 = \boxed{171}$ .

10. **[\$8, 18]** Anthony draws a regular hexagon  $ABCDEF$  and makes a cut of length 13 through the center. The heights from  $A$  and  $C$  to the cut are in the ratio 5 : 7. Find the side length of the hexagon.



**Solution:**  $\boxed{\frac{7\sqrt{39}}{6}}$

Refer to the points labeled below.



Note that  $AP \parallel OC$ , so  $\angle ABO = \angle BOC$ , so the right triangles with hypotenuses  $AP$  and  $OC$  are similar. Thus we can set  $AP = 5x$  and  $CO = 7x$ . But  $OC$  is the side length of the hexagon.

The horizontal component of the cut has length  $5x + 5x - 7x = 3x$ , and the vertical component is  $7\sqrt{3}x$ . So by the Pythagorean Theorem the length of the cut is

$$13 = x\sqrt{3^2 + (7\sqrt{3})^2} = x\sqrt{156}.$$

Solving, we get  $x = \frac{13}{\sqrt{156}} = \frac{\sqrt{39}}{6}$ . The answer is then  $7x = \boxed{\frac{7\sqrt{39}}{6}}$ .

11. **[\$9, 21]** Find the remainder when  $\frac{669!}{336! \cdot 334!}$  is divided by 67.

**Solution:**  $\boxed{59}$

Let  $\nu_p(n)$  denote the largest  $k$  for which  $p^k$  divides  $n$ . We will use Wilson's Theorem. Notice that

$$\frac{669!}{\nu_{67}(669!)} \equiv 1 \cdot 2 \cdots 66 \cdot \frac{67}{67} \cdot 68 \cdots 133 \cdot \frac{134}{67} \cdot 135 \cdots 669.$$

Then

$$\equiv 66! \cdot 1 \cdot 66! \cdot 2 \cdots \equiv (66!)^9 \cdot 9! = (-1)^9 \cdot 9!.$$

Similarly,

$$\frac{336!}{\nu_{67}(336!)} \equiv 1 \cdot 2 \cdots 66 \cdot \frac{67}{67} \cdot 68 \cdots 133 \cdot \frac{134}{67} \cdot 135 \cdots 336 \equiv (-1)^5 \cdot 5! \cdot 1 = (-1) \cdot 5!,$$

and then

$$\frac{334!}{\nu_{67}(334!)} \equiv (-1)^4 \cdot 4!.$$

So our desired value is  $\frac{-9!}{-5! \cdot 4!} \equiv 126 \equiv \boxed{59}$ .

12. **[\$9, 22]** Suppose  $a, b, c$  are positive integers satisfying  $\gcd(a, b) = 6$ ,  $\gcd(a, c) = 15$ , and  $\text{lcm}(b, c) = 180$ . Find the minimum possible value of  $a + b + c$ .

**Solution:**  $\boxed{81}$

We will consider  $\nu_2, \nu_3$ , and  $\nu_5$  of each equation separately.

We have  $\nu_2(6) = 1$ ,  $\nu_2(15) = 0$ , and  $\nu_2(180) = 2$ . These are all distinct, so  $a, b$ , and  $c$  have distinct  $\nu_2$  values (specifically 0, 1, and 2). Clearly  $a$  and  $b$  have  $\nu_2$  value greater than 0, so  $\nu_2(c) = 0$ . Then we see the only solution is  $\nu_2(a) = 1$  and  $\nu_2(b) = 2$ .

For  $\nu_3$ , the values are 1, 1, and 2. The this happens when two of them have  $\nu_3$  value 1, and the other has  $\nu_3$  value 2. The one with  $\nu_3$  value 2 must either be  $b$  or  $c$ .

For  $\nu_5$ , the values are 0, 1, and 1. This forces  $\nu_5(b) = 0$ . The values of  $\nu_5(a)$  and  $\nu_5(c)$  are at least 1, and setting them equal to 1 works. Since we are trying to minimize  $a, b, c$ , this is optimal.

Thus  $a = 2 \cdot 3 \cdot 5$ . We see  $b$  is 4 times 3 or 9, and  $c$  is 5 times the other one of 3 and 9. Their sum is minimized when  $b = 4 \cdot 9$  and  $c = 5 \cdot 3$ . The answer is then  $30 + 36 + 15 = \boxed{81}$ .

13. **[\$10, 26]** Suppose  $x$  is a real number. Find the sum of all possible values of  $x^2$ , given that

$$\frac{x^3 + 2}{5} = \sqrt[3]{5x - 2}.$$

**Solution:**  $\boxed{10}$

Note that  $\frac{x^3+2}{5}$  and  $\sqrt[3]{5x-2}$  are inverse functions, so we can check that they only intersect when  $\frac{x^3+2}{5} = \sqrt[3]{5x-2} = x$ . Thus, our solutions are from the polynomial  $x^3 - 5x + 2$ . Testing yields that  $x = 2$  is a root of this; dividing it out yields that:

$$x^3 - 5x + 2 = (x - 2)(x^2 + 2x - 1) = (x - 2)(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$$

Hence,  $S = \{2, -1 + \sqrt{2}, -1 - \sqrt{2}\}$  (testing yields that these are indeed all solutions). Hence, our answer is  $4 + 3 - 2\sqrt{2} + 3 + 2\sqrt{2} = \boxed{10}$ .

14. **[\$10, 27]** Aryan arranges  $20^3$  unit cubes into a large  $20 \times 20 \times 20$  cube. Find the number of ordered pairs  $(A, B)$  of two distinct unit cubes for which every contiguous  $4 \times 4 \times 4$  of unit cubes that contains  $A$  also contains  $B$ .

**Solution:**  $\boxed{24768}$

Replace 20 by  $n$  and 4 by  $k$ . For the  $\ell$ th row, define  $R(\ell)$  to be the number of rows which are contained in any  $4 \times 4$  grid containing a cell in the  $\ell$ th row. Clearly  $R(1) = R(n) = k$ ,  $R(2) = R(n-1) = k-1$ ,  $R(3) = R(n-2) = k-2$ . This count keeps decreasing until it is just 1. Define  $C(\ell)$  for columns, and  $D(\ell)$  for the depth in a similar manner. The values of  $C$  and  $D$  are exactly the same as  $R$ .

For a cell  $A$  with coordinates  $(i, j, k)$ , the number of possible  $B$  is  $R(i) \cdot C(j) \cdot D(k)$ . Our answer is then

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n R(i)C(j)D(k) = \left( \sum_{i=1}^n R(i) \right)^3.$$

So it suffices to compute the sum of  $R(i)$  taken over all  $i$ . For  $n > 2k$ , this is

$$k + (k-1) + \cdots + 1 + 1 \cdot (n-2k) + 1 + 2 + \cdots + k = n + k^2 - k.$$

But note that this will count the  $n^3$  pairs  $(A, A)$ . So our answer is  $(n + k^2 - k)^3 - n^3$  which gives  $(20 + 4^2 - 4)^3 - 20^3 = \boxed{24768}$ .

15. **[\$11, 31]** Five athletes are ranked based on their speed and strength (where 1 is the best, and 5 is the worst). They all know each other's ranks, and make the following truthful statements:

- Amara: I have the same rank in both categories.
- Billy: I'm first in something.
- Cris: Dante is the only person stronger and faster than me.
- Dante: I'm not first in anything.
- Elena: I'm the weakest (in strength).

Let  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  be the product of the speed rank and strength rank for Amara, Billy, Cris, Dante, and Elena, respectively. Find  $A + 2B + 3C + 4D + 5E$ .

**Solution:**  $\boxed{94}$

Refer to each person by the first letter of their first name. It helps to think about the order as a  $5 \times 5$  grid where if a person is in the  $i$ th row and  $j$ th column, they are rank  $i$  in one category (speed or strength) and  $j$  in the other.

$A$  tells us they are on the diagonal. Additionally  $B$  and  $E$  tell us that  $A$  cannot be in the top left or bottom right.

$D$  tells us they cannot be in the first column or row, and  $C$  is below and to the right of  $D$ . Also  $B$  cannot be above and to the left of  $D$ , since then they would be above and to the left of  $C$ . Then the relative positions of  $B$ ,  $C$ , and  $D$  are:

		$B$
$D$		
	$C$	

In particular,  $C$  is no better than the third row and third column ( $D$  and  $B$  are above it, and  $D$  is to the left but is not first place).

If we try to place  $A$  in the second row and second column, then  $C$  is worse than  $A$  in both categories. If we try to place  $A$  in the third row and third column the same thing happens. Thus  $A$  is in the 4th row and 4th column.

The first column and first row are off limits for  $C$  and  $D$ . They must be in the (second row/third row) and (second column/second column). The only way to orient them is  $D$  in the second row/column, and  $C$  in the third row/column. Finally  $E$  and  $B$  are in the bottom left and top right corners. Here is the final table:

				$B$
	$D$			
		$C$		
			$A$	
$E$				

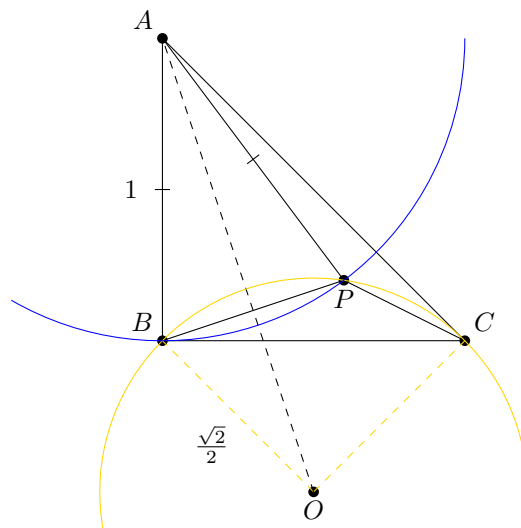
Then the answer is

$$1 \cdot 4 \cdot 4 + 2 \cdot 5 \cdot 1 + 3 \cdot 3 \cdot 3 + 4 \cdot 2 \cdot 2 + 5 \cdot 1 \cdot 5 = \boxed{94}.$$

16. **[\$11, 32]** Let  $ABC$  be an isosceles triangle with  $\angle B = 90^\circ$ . Point  $P$  lies inside  $ABC$  with  $\angle BPC = 135^\circ$  and  $AP = AB = 1$ . Find  $BP$ .

**Solution:**  $\boxed{\frac{\sqrt{10}}{5}}$

Let  $O$  be the circumcenter of  $BPC$ .



Then note that  $BOC$  is an isosceles right triangle with  $\angle BOC = 90^\circ$ . Additionally  $P$  is the point lying on this circle as well as the circle centered at  $A$  with radius 1.

We have  $\angle ACO = 90^\circ$ , so we can compute  $AO^2 = (\sqrt{2})^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{5}{2}$ .

On one hand the area of quadrilateral  $ABOP$  is  $\frac{1}{2}BP \cdot AO = BP \cdot \sqrt{\frac{5}{8}}$ . But we can also calculate it using  $AB = 1$ ,  $BO = \frac{\sqrt{2}}{2}$ , and  $\angle ABO = 135^\circ$  using sine area formula as  $1 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$ . Thus  $BP \cdot \sqrt{\frac{5}{8}} = \frac{1}{2}$ , so  $BP = \boxed{\frac{\sqrt{10}}{5}}$ .

17. **[\$12, 36]** Find

$$\sum_{n=1}^{30} \binom{n}{\lceil \frac{n+1}{2} \rceil} = \binom{1}{1} + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16}.$$

**Solution:**  $\boxed{\binom{31}{15} - 1}$

Add 1 to the sum and combine with 1 with  $\binom{1}{1}$  to get  $\binom{2}{1}$ . Then by repeated applications of Pascal's Identity

and  $\binom{n}{k} = \binom{n}{n-k}$  we get

$$\begin{aligned}
 \binom{2}{1} + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} &= \binom{3}{2} + \binom{3}{2} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &= \binom{3}{1} + \binom{3}{2} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &= \binom{4}{2} + \binom{4}{3} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &= \binom{5}{3} + \binom{5}{3} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &= \binom{5}{2} + \binom{5}{3} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &= \binom{6}{3} + \binom{6}{4} + \cdots + \binom{28}{15} + \binom{29}{15} + \binom{30}{16} \\
 &\quad \dots \\
 &= \binom{30}{15} + \binom{30}{16} \\
 &= \binom{31}{15}.
 \end{aligned}$$

Thus the answer is  $\boxed{\binom{31}{15} - 1}$ .

18. **[\$12, 38]** Find the sum of all positive integers  $n < 210$  for which  $\gcd(n^n - 1, 210) = 5$ .

**Solution:**  $\boxed{378}$

First consider modulo 2. Then  $n$  cannot be odd, so  $n$  is even. Next consider modulo 3. If  $n$  is nonzero mod 3, then  $n^{\text{even}} \equiv 1 \pmod{3}$ , thus  $n$  must be divisible by 3.

Similarly for mod 7 we see if  $n \not\equiv 0 \pmod{7}$  then  $n^n \equiv n^{\text{multiple of } 6} \equiv 1 \pmod{7}$ . Thus  $7 \mid n$ . Combining all of this we get  $42 \mid n$ , so let  $n = 42k$ .

Modulo 5 we have  $(42k)^{42k} \equiv (2k)^{2k} \pmod{5}$ . Checking  $k = 1, 2, 3$ , and 4 we see  $k = 2, 3$ , and 4 work. So the answer is  $42 \cdot (2 + 3 + 4) = \boxed{378}$ .

19. **[\$13, 43]** Let  $a, b$ , and  $c$  be real numbers satisfying

$$a + ab = 4, \quad 2b + bc = 5, \quad 3c + ca = 6.$$

Find the sum of all possible values of  $(abc)^2$ .

**Solution:**  $\boxed{1465}$

Set  $x = abc$ . Multiply all the equations to get

$$\begin{aligned}
 120 &= a(b+1)b(c+2)c(a+3) \\
 &= x(a+3)(b+1)(c+2) \\
 &= x(x+6+2a+2ab+6b+3bc+3c+ca) \\
 &= x(x+6+2(a+ab)+3(2b+bc)+1(c+ca)) \\
 &= x(x+6+2 \cdot 4+3 \cdot 5+1 \cdot 6) \\
 &= x(x+6+8+15+6) \\
 &= x(x+35).
 \end{aligned}$$

Thus  $x^2 + 35x - 120 = 0$ . The sum of all possible values of  $x^2$  is  $35^2 + 120 \cdot 2 = \boxed{1465}$ .

20. **[\$13, 45]** Find the sum of all positive integers  $n < 455$  for which the remainders when  $n$  is divided by 5, 7, and 13 all sum to 11.

**Solution:**  $\boxed{7945}$

Note that  $455 = 5 \cdot 7 \cdot 13$ . For each residue mod 455, consider its residues mod 5, 7, and 13 separately. If  $n$  has remainders  $a$ ,  $b$ , and  $c$  mod 5, 7, and 13, respectively, then  $454 - n$  has remainders  $4 - a$ ,  $6 - b$ , and  $12 - c$ . The sum of the remainders for  $n$  is  $a + b + c$ , and for  $454 - n$  it is  $22 - (a + b + c)$ . Thus, if  $a + b + c = 11$ , then  $22 - (a + b + c) = 11$  as well. So if  $n$  works, then so does  $454 - n$ .

So it suffices to compute the number of  $n$  with this property. We have 5 choices for a residue mod 5, and 7 choices for a residue mod 7. This will uniquely determine a residue mod 13, and this will always be possible as the sum of the first two residues is at most  $4 + 6 = 10$ . Thus there are 35 such  $n$  (this is odd because  $\frac{454}{2}$  works). The answer is then

$$454 \cdot \frac{35}{2} = 227 \cdot 35 = \boxed{7945}.$$

21. **[\$14, 51]** Let  $a$ ,  $b$ , and  $c$  be positive real numbers such that

$$a^2 + b^2 - ab = 49, \quad b^2 + c^2 + bc\sqrt{3} = 225, \quad a^2 + c^2 + ac\sqrt{3} = 400.$$

Find the value of  $a\sqrt{900 - b^2} + b\sqrt{1600 - a^2}$ .

**Solution:**  $\boxed{336}$

Note that our equations can be thought of as

$$a^2 + b^2 - 2ab \cos(60^\circ) = 49$$

$$b^2 + c^2 - 2bc \cos(150^\circ) = 225$$

$$a^2 + c^2 - 2ac \cos(150^\circ) = 400$$

The above are law of cosine expressions for a triangle of side lengths 7, 15, and 20, with a point in the triangle satisfying the angle conditions.

Let  $\triangle ABC$  be the triangle with  $AB = 7$ ,  $BC = 15$ , and  $AC = 20$ , and let  $P$  be the point of interest. Note that

$$\begin{aligned} a\sqrt{900 - b^2} + b\sqrt{1600 - a^2} &= 1200 \left( \frac{a\sqrt{900 - b^2}}{1200} + \frac{b\sqrt{1600 - a^2}}{1200} \right) \\ &= 1200 \left( \frac{a}{40} \frac{\sqrt{900 - b^2}}{30} + \frac{b}{30} \frac{\sqrt{1600 - a^2}}{40} \right) \\ &= 1200 \sin \left( \arcsin \left( \frac{a}{40} \right) + \arcsin \left( \frac{b}{30} \right) \right). \end{aligned}$$

Then, note by law of sines that

$$\frac{20}{\sin(150^\circ)} = \frac{a}{\sin(\angle PCA)} \implies \sin(\angle PCA) = \frac{a}{40} \implies \angle PCA = \arcsin \left( \frac{a}{40} \right).$$

Similarly,

$$\frac{15}{\sin(150^\circ)} = \frac{b}{\sin(\angle PCB)} \implies \sin(\angle PCB) = \frac{b}{30} \implies \angle PCB = \arcsin \left( \frac{b}{30} \right).$$

So,

$$\arcsin \left( \frac{a}{40} \right) + \arcsin \left( \frac{b}{30} \right) = \angle PCA + \angle PCB = \angle ACB.$$

Using law of cosines, we have that

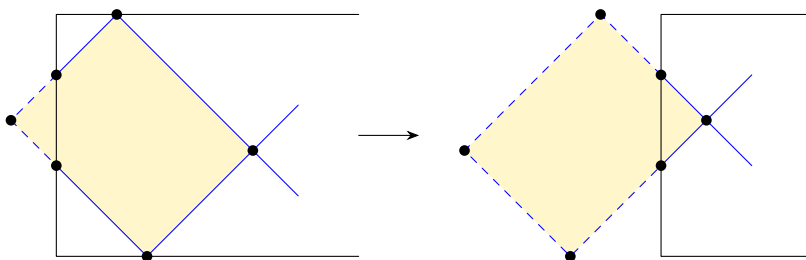
$$49 = 225 + 400 - 600 \cos(\angle ACB) \implies \cos(\angle ACB) = \frac{24}{25}.$$

Since  $\angle ACB$  is acute,  $\sin(\angle ACB) = \frac{7}{25}$ , which gives our answer as  $1200 \cdot \frac{7}{25} = \boxed{336}$ .

22. **[\$14, 53]** There are mirrors on sides  $AB$  and  $CD$  of rectangle  $ABCD$  facing the interior, with  $AD = 16$  and  $AB = 25$ . Seven points split  $AD$  into 8 segments of length 2, and a laser is shot from each point. Each laser forms a  $45^\circ$  angle with  $AD$  and goes towards  $AB$  or  $CD$ , each with probability  $\frac{1}{2}$ . Find the the expected number of times two lasers intersect before hitting side  $BC$ .

**Solution:**  $\boxed{\frac{129}{4}}$

Observe that once two lasers intersect, they will intersect again after they travel 16 units horizontally. Imagine extending the rectangle to the left of segment  $AD$ , and continuing the lasers in that direction. Using this, we see that by reversing the directions of the lasers, the horizontal distance traveled until they intersect changes from  $k$  to  $16 - k$ .



Thus arbitrarily fix the direction of two lasers. They must intersect in the first  $16 \times 16$  square. In the remaining  $9 \times 16$  square, there are three options.

- They intersect before 8. Then reversing the direction of the two lasers makes it so that they don't intersect.
- They intersect at 8. Then reversing the directions makes it so that they still intersect.
- They don't intersect. Then reversing the directions of the two lasers makes it so they do intersect.

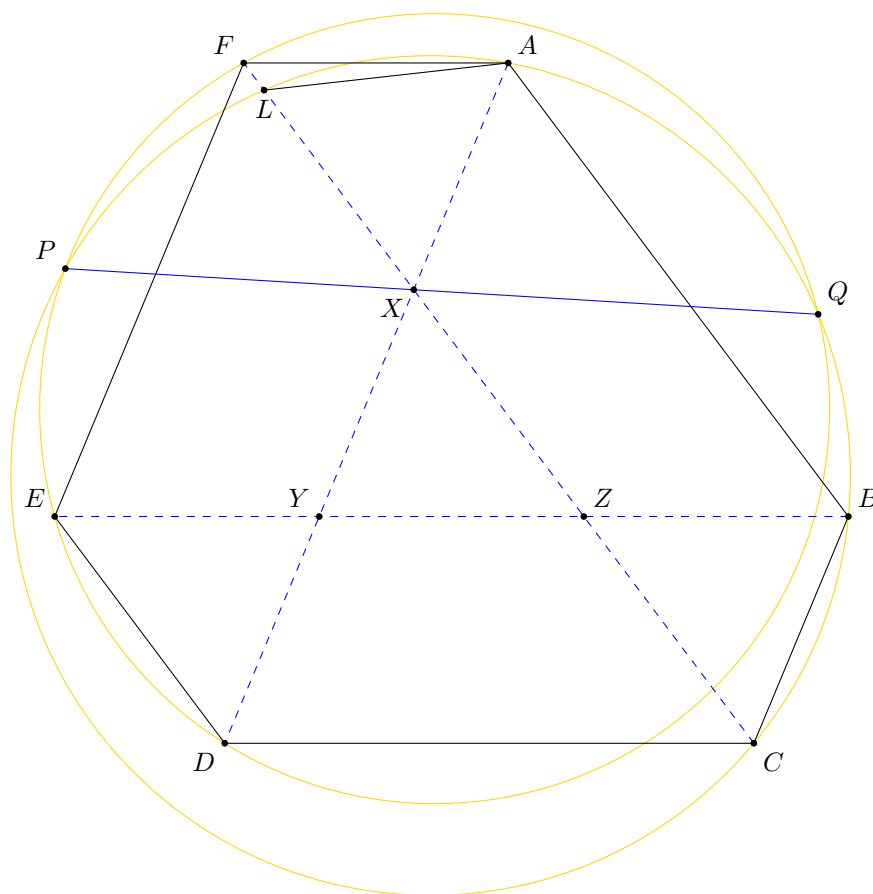
In this way, we can almost creating a pairing between a configuration and the configuration obtained by reversing each of the lasers. If the lasers intersect  $k$  times, the reversed lasers intersect  $3 - k$  times.

The exception is the case where they intersect at 8. This can only happen if their starting positions are exactly 8 units apart. There are only 3 pairs of starting points which are 8 units apart. Accounting for this case changes the expected value of these pairs from  $\frac{3}{2}$  to  $\frac{7}{4}$ . In total the answer is then

$$\left( \binom{7}{2} - 3 \right) \cdot \frac{3}{2} + 3 \cdot \frac{7}{4} = \boxed{\frac{129}{4}}.$$

23. **[\$15, 59]** Let  $ABCDEF$  be a hexagon with area 100 and opposite sides parallel. Suppose  $AB = 2DE$ ,  $CD = 2FA$ ,  $EF = 2BC$ . The circumcircles of  $ABC$  and  $DEF$  intersect at points  $P$  and  $Q$ . Suppose  $PQ \parallel FA$ , and line  $PQ$  splits  $ABCDEF$  into two polygons. Find the area of the smaller polygon.

**Solution:**  $\boxed{\frac{300}{13}}$



Motivated by the length conditions, we construct the diagonals  $AD$ ,  $BE$ , and  $CF$ , intersecting at  $X$ ,  $Y$ , and  $Z$  as pictured. Then note that  $\triangle AXF \sim \triangle DXC$  with ratio  $1 : 2$  and  $\triangle BZC \sim \triangle EZF$  with ratio  $1 : 2$  as well. Then we have

$$\frac{1}{2} = \frac{XF}{XC} = \frac{CZ}{CX}$$

Since  $F$ ,  $X$ ,  $Z$ , and  $C$  are collinear, we see that  $X$  and  $Z$  trisect segment  $FC$ . From this we see that hexagon  $ABCDEF$  can be drawn by starting with triangle  $XYZ$  and reflecting each vertex over the other 2.

**Claim:** The radical axis of  $(ABC)$  and  $(DEF)$  passes through  $X$ .

*Proof.* We can compute the power of  $X$  with respect to  $(ABC)$ . First let  $(ABC) \cap CF = L$ . Then note that  $LABC$  is an isosceles trapezoid, and the power of  $X$  with respect to  $(ABC)$  is  $XL \cdot XC$ . But note that  $LAX$  is isosceles, so we can rewrite  $XL$  as  $2 \cdot AX \cdot \sin(\angle X) = 2XY \sin \angle X$ . So the power is

$$XL \cdot XC = 4 \cdot XY \cdot XZ \cdot \sin \angle X.$$

Repeating these steps on  $(DEF)$  gives the same power, as desired. ■

To conclude, we can use areas ratios to see  $XYZ$  is  $\frac{1}{13}$  of the area of  $ABCDEF$ . Since  $PQ \parallel FA$ , we see it splits  $ABCDEF$  into two parts, one of which has 3 copies of  $XYZ$  and another which has 10. So the answer is  $\frac{3}{13} \cdot 100 = \boxed{\frac{300}{13}}$ .

24. **[\$15, 62]** Let  $\lfloor x \rfloor$  denote the greatest integer less than or equal to  $x$ . Find the remainder when

$$\left[ \left( 15 + 6\sqrt{6} + 10\sqrt{2} + 8\sqrt{3} \right)^{1919} + \left( 15 + 6\sqrt{6} - 10\sqrt{2} - 8\sqrt{3} \right)^{1919} \right]$$

is divided by 100.

**Solution:**  $\boxed{59}$

We can consider the expression for general exponent  $k$ . Note that

$$15 + 6\sqrt{6} + 10\sqrt{2} + 8\sqrt{3} = (3 + 2\sqrt{2})(5 + 2\sqrt{6})$$

and

$$15 + 6\sqrt{6} - 10\sqrt{2} - 8\sqrt{3} = (3 - 2\sqrt{2})(5 + 2\sqrt{6}).$$

Now, consider  $(3 + 2\sqrt{2})(5 - 2\sqrt{6})$  and  $(3 - 2\sqrt{2})(5 - 2\sqrt{6})$ . Noting that  $1.5 > \sqrt{2} > 1.4$  and  $2.45 > \sqrt{6} > 2.44$ , we can verify that  $(3 + 2\sqrt{2})(5 - 2\sqrt{6}) < .72$  and  $(3 - 2\sqrt{2})(5 - 2\sqrt{6}) < 0.024$ .

Now, consider

$$S_k = \left( (3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k \right) \left( (5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k \right),$$

which equals

$$\begin{aligned} & (15 + 6\sqrt{6} + 10\sqrt{2} + 8\sqrt{3})^k + (15 + 6\sqrt{6} - 10\sqrt{2} - 8\sqrt{3})^k \\ & + (15 - 6\sqrt{6} + 10\sqrt{2} - 8\sqrt{3})^k + (15 - 6\sqrt{6} - 10\sqrt{2} + 8\sqrt{3})^k. \end{aligned}$$

For all  $k \in \mathbb{N}$ , note that  $(15 - 6\sqrt{6} + 10\sqrt{2} - 8\sqrt{3})^k + (15 - 6\sqrt{6} - 10\sqrt{2} + 8\sqrt{3})^k < 1$ , so we can just find the last two digits of  $S_k$  and just subtract by 1 to get our answer.

Now, we essentially want  $S_{1919} \pmod{100}$ .

Note that  $(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k$  is an explicit solution to  $a_{k+2} = 6a_{k+1} - a_k$  with  $a_0 = 2$  and  $a_1 = 6$ , and  $(5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k$  is an explicit solution to  $b_{k+2} = 10b_{k+1} - b_k$  with  $b_0 = 2$  and  $b_1 = 10$ . Via quick examination, we can verify that both sequences will have even terms, so  $S_{1919} \equiv 0 \pmod{4}$ .

To check for  $S_{1919} \pmod{25}$ , note first that

$$\begin{aligned} (5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k &= 2 \cdot \sum_{k=0}^{959} \binom{1919}{2k} (5)^{1919-2k} (2\sqrt{2})^{2k} \\ &= 2 \cdot \sum_{k=0}^{959} \binom{1919}{2k} (5)^{1919-2k} (24)^k \\ &\equiv 2 \cdot \binom{1919}{1} 5(24^{959}) \pmod{25}. \end{aligned}$$

where the last equality follows since every other term in the sum is divisible by 25. Evidently, this sum is divisible by 5, meaning that we can divide it out and consider our remaining product modulo 5:

$$2 \cdot \binom{1919}{1} (24^{959}) \equiv 2(-1)(-1) \equiv 2 \pmod{5}.$$

For the  $(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k$ , we use the recursion modulo 5, which yields the sequence

$$2, 1, 4, 3, 4, 1, 2, 1$$

which repeats every 6. This yields  $a_{1919} \equiv 1 \pmod{5}$ . Hence,

$$\left( (3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k \right) \cdot \left( \frac{(5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k}{5} \right) \equiv 2 \pmod{5},$$

which implies

$$\left( (3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k \right) \left( (5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k \right) \equiv 10 \pmod{25}.$$

Hence, using Chinese Remainder Theorem,

$$\left( (3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k \right) \cdot \left( (5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k \right) \equiv 60 \pmod{100},$$

which means our answer is  $\boxed{59}$ .